J. Fluid Mech. (1985), *vol.* 155, *pp.* 359–379
Printed in Great Britain

Conically similar viscous flows. Part 3. Characterization of axial causes in swirling flow and the one-parameter flow generated by a uniform half-line source of kinematic swirl angular momentum

By R. PAULL AND A. F. PILLOW
Department of Mathematics, University of Queensland

(Received 27 July 1984 and in revised form 31 December 1984)

In Part 1 of this series conservation principles for ring circulation and kinematic swirl angular momentum were developed for general axisymmetric incompressible viscous flow. These principles were then used to classify the four independent axial causes of swirl-free conically similar viscous flow. Part 2 provided a detailed analysis of the one-parameter swirl-free flows that are generated by each one of the axial singularities acting alone. The present paper extends, to swirling flow, the description of the axial singularities that drive axisymmetric viscous flow. In the special case of conically similar viscous flow, two independent half-line sources of swirl angular momentum suffice to complete the set of axial singularities that can generate such swirling flows. The individual strengths of the six independent axial causes provide a complete characterization of all conically similar viscous flows that can be generated in this way. This Part 3 completes the task of analysing in detail the independent one-parameter flows generated by axial causes by studying the flow caused by uniform production of kinematic swirl angular momentum on a half-axis. This flow demonstrates how swirl may induce an axial half-plane flow. For large swirl circulation strengths, swirl angular momentum diffuses and convects so as to fill slightly more than half the space with an almost constant density of swirl angular momentum. A well-developed internal boundary layer, in the form of an outward radial jet, then separates this region from one in which the flow is almost irrotational. The jet entrains two impinging convection fields. The angular location of the jet is determined by relating the axial component of moment of whirl produced at the origin to the strength of the swirling circulation singularity on the axis.

1. Introduction

Conservation of ring circulation and of volume were used in Part 1 (Pillow & Paull 1985, hereinafter referred to as I) of this series of papers to show that there are just three different independent types of axial singularities that generate swirl-free conically similar viscous flows when no physical boundaries are present. One such flow type arises when volume sources are uniformly distributed along a half-axis. The remaining two types arise from production of ring circulation either at the origin, or along the whole axis with an appropriate line density which is antisymmetric about the origin. These latter two types of flow have the strength of their singularities measured by the independently conserved radial and transverse rates of discharge of the axial component of moment of whirl. Details of this classification were given in §4 of I. Part 2 (Paull & Pillow 1985, hereinafter referred to as II) presented the flows attributable to these distinct causes and elucidated the roles of the individual diffusion and convection terms in the flux vector $\boldsymbol{J}/2\pi$ for ring circulation.

In swirling conically similar viscous flow, as pointed out in §2 of I, not only does $\boldsymbol{J}$ require modification in order to account for swirl, but a new conservation principle for kinematic swirl angular momentum comes into play. Dimensional analysis shows that a further independent axial singularity that can generate such flows is provided by uniform production of kinematic swirl angular momentum on a half-axis. Independent strengths for these singularities on each half-axis then provide sufficient extra conditions for the sixth-order system of coupled ordinary differential equations I (3.4), I (3.5) and I (3.6) to become well-posed, when the other four swirl-free singularities are specified. (Here I (3.4), for example, refers to equation (3.4) of Part 1.)

This present paper is concerned with classifying swirling conically similar viscous flows in terms of their axial causes and with the final independent problem in the set of six one-parameter solutions generated by the independent axial singularities. The latter is the swirling flow in which kinematic swirl angular momentum (i.e. swirl angular momentum per unit mass) is produced uniformly along only one half-axis in an infinite space. Several authors (Gol'dshtik 1960; Hoffmann 1974; Kidd & Farris 1968; Schwiderski 1969; Serrin 1972; Whang 1968; Wygnanski 1970; Yih *et al.* 1982) have investigated swirling causes with conical boundaries. Coupling effects make it difficult to characterize the parameters in such flows directly in terms of the rate of production of conserved quantities, particularly when no-slip boundary conditions are involved. Here in this paper there is no boundary and no production of volume on the axis. It is the strength of the swirling singularity alone that drives the motion. (Coupling of axial causes and conically similar production of conserved quantities on cones will be examined in later papers in terms of conservation principles.) For small-strength kinematic swirl angular-momentum production the induced axial half-plane flow is away from the half-space in which the uniform axial distribution of kinematic swirl angular-momentum sources lies. As the strength of production increases, a reversed-flow region neighbouring the free half-axis appears and develops so that, when the strength of production is large, the separatrix is directed almost normal to the axis of symmetry. To first order, kinematic swirl angular momentum is then confined to the side of this separatrix containing the kinematic swirl angular-momentum sources. An internal boundary layer in the form of a transition between swirl-free irrotational flow and constant-swirl-circulation flow is then present at this separatrix and there is a strong radially outwards flow in the neighbourhood of the internal interface.

The above problem is formulated mathematically in §4, and the solutions for small and large values of the strength of production of kinematic swirl angular momentum are given in §§5 and 6 respectively. The new features of diffusion, convection and viscous convection of kinematic swirl angular momentum and the flow of ring circulation along the vortex tubes, which were described in §2 of I, are illustrated and discussed in §7. This affords a more detailed understanding of how swirl may induce an axial half-plane flow. Computer-generated solutions and visualizations of the flow field supplement the discussion.

In swirling flow the conservation principle for the axial component of moment of whirl, which was formulated in I (4.7) and modified in I (4.12), requires further modification if the sources of kinematic swirl angular momentum and of the axial component of moment of whirl are to remain uncoupled. The decoupling of these flow singularities provides the specification of axial causes in swirling conically similar viscous flows described in §3 of this paper.

2. Conservation principles in swirling axisymmetric flow

It was shown in §1 of I that the velocity field $\boldsymbol{q}(\boldsymbol{r},t)$ and the vorticity field $\boldsymbol{\omega}(\boldsymbol{r},t)$ can be broken into axial half-plane and azimuthal components in the form

$$\boldsymbol{q} = \boldsymbol{u} + \frac{T}{\sigma}\hat{\boldsymbol{\phi}} \tag{2.1}$$

and

$$\boldsymbol{\omega} = \operatorname{curl}\boldsymbol{q} = \boldsymbol{\Omega} + \sigma l\hat{\boldsymbol{\phi}}, \tag{2.2}$$

in terms of cylindrical polar coordinates (x,σ,ϕ). Since both $\boldsymbol{u}$ and $\boldsymbol{\Omega}$ are solenoidal, they can be described in terms of a stream function ψ and the swirl circulation $2\pi T$:

$$\boldsymbol{u} = \frac{1}{\sigma}[\psi_\sigma\hat{\boldsymbol{x}} - \psi_x\hat{\boldsymbol{\sigma}}] \tag{2.3}$$

and

$$\boldsymbol{\Omega} = \frac{1}{\sigma}[T_\sigma\hat{\boldsymbol{x}} - T_x\hat{\boldsymbol{\sigma}}]. \tag{2.4}$$

In terms of ψ, l is given by

$$\psi_{xx} - \frac{1}{\sigma}\psi_\sigma + \psi_{\sigma\sigma} = -\sigma^2 l. \tag{2.5}$$

Conservation of ring circulation (I (2.3)) requires that

$$\frac{\partial l}{\partial t} + \operatorname{div}\boldsymbol{J} = -4\pi\nu l\delta(\boldsymbol{\sigma}), \tag{2.6}$$

where, by I (2.9), the ring circulation flux vector $\boldsymbol{J}/2\pi$ is given by

$$\boldsymbol{J} = l\boldsymbol{u} - \frac{T}{\sigma}\frac{\boldsymbol{\Omega}}{\sigma} - 2l\boldsymbol{q}_0 - \nu\nabla l. \tag{2.7}$$

Here $\boldsymbol{q}_0 = \nu\hat{\boldsymbol{\sigma}}/\sigma$, $\operatorname{div}\boldsymbol{q}_0 = 2\pi\nu\delta(\boldsymbol{\sigma})$, and $l/2\pi$ is the ring-circulation volume density. In swirling flow, axial half-plane ring circulation is produced by rotation of vortex tubes as a result of a gradient in the swirl circulation. It was shown in §2 of I that the integrated effect of this production could be incorporated into the flux vector $\boldsymbol{J}$ as a new term $-T\boldsymbol{\Omega}/\sigma^2$, which described the flow of ring circulation along the vortex lines of the axial half-plane vorticity $\boldsymbol{\Omega}$.

Conservation of kinematic swirl angular momentum (I (2.12)) requires that

$$\frac{\partial T}{\partial t} + \operatorname{div}\boldsymbol{K} = 0, \tag{2.8}$$

where, by I (2.13), the kinematic swirl angular-momentum flux vector $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = T\boldsymbol{u} + 2T\boldsymbol{q}_0 - \nu\nabla T \tag{2.9}$$

and T is the kinematic swirl angular-momentum volume density. The convection term $T\boldsymbol{u}$ and the diffusion term $-\nu\nabla T$ in $\boldsymbol{K}$ are completely analogous to the corresponding terms in the ring-circulation flux vector $\boldsymbol{J}/2\pi$. However, the viscous convection term $2T\boldsymbol{q}_0$ in $\boldsymbol{K}$ now acts in the opposite direction to the corresponding term in $\boldsymbol{J}$.

The conservation principle I (4.7) for the axial component of moment of whirl was derived for swirl-free flow in §4 of I in order to measure singular production of ring circulation on the axis of symmetry in conically similar viscous flows. The corresponding formal generalization of I (4.7) for swirling flow can also be derived directly from (2.6) and (2.7) after multiplication by σ^2 and reformulation as a conservation equation. It is

$$\frac{\partial m}{\partial t}+\operatorname{div}[m(\boldsymbol{u}+2\boldsymbol{q}_0)-\nu\nabla m-T\boldsymbol{\Omega}+\{2\boldsymbol{q}\boldsymbol{q}-(\boldsymbol{q}\cdot\boldsymbol{q})\boldsymbol{I}\}\cdot\hat{\boldsymbol{x}}]=0, \tag{2.10}$$

where m is the volume density of the axial component of moment of whirl. Here $m=\sigma^2 l$ and

$$\boldsymbol{q}=u\hat{\boldsymbol{x}}+v\hat{\boldsymbol{\sigma}}+\frac{T}{\sigma}\hat{\boldsymbol{\phi}}. \tag{2.11}$$

It was pointed out in §4 of I that if volume sources of variable strength are distributed along the axis and generate the irrotational velocity field $u_{\mathrm{p}}\hat{\boldsymbol{x}}+v_{\mathrm{p}}\hat{\boldsymbol{\sigma}}$ then the flux vector in (2.10) requires modification if it is to remain independent of the flow induced by the volume sources. Subtraction of the solenoidal flux vector

$$(u_{\mathrm{p}}^2-v_{\mathrm{p}}^2)\hat{\boldsymbol{x}}+2u_{\mathrm{p}}v_{\mathrm{p}}\hat{\boldsymbol{\sigma}}, \tag{2.12}$$

given in I (4.10), ensures that the modified flux vector I (4.12) for the axial component of moment of whirl vanishes in potential flow, since the non-viscous contributions to it now only amount to interactions between the ring circulation and potential flow fields or the ring circulation field and itself (I (4.14)).

When swirl is present, there is, as well, distributed production of ring circulation occurring as a result of rotation and stretching of the vortex tubes. This distributed production is replaced by the additional flux terms $-T^2\hat{\boldsymbol{x}}/\sigma^2$ and $-T\boldsymbol{\Omega}$ which appear in (2.10). Even when swirl is present, the production of the axial component of moment of whirl can thus still be confined to the axis of symmetry and $|x|=\infty$. Conceptually, the strengths of these fluxes suffice to determine the axial half-plane flow.

In (2.10) the flux term $-T^2\hat{\boldsymbol{x}}/\sigma^2$ formally measures the integrated effect of the distributed production of the axial component of moment of whirl, while $-T\boldsymbol{\Omega}$ represents the flux of the appropriate moment of ring circulation along the vortex tubes. The former requires modification since it predicts a line vortex, with constant swirl circulation $2\pi c$, has a flux $-c^2\hat{\boldsymbol{x}}/\sigma^2$ of the axial component of moment of whirl, despite an absence of ring circulation in the flow. This anomaly is a result of non-specification of the causes as $|x|\rightarrow\infty$ and may be corrected if it is noted that exactly the same term $-c^2\hat{\boldsymbol{x}}/\sigma^2$ would arise in the flux vector of (2.10) if the flow were irrotational and generated by a uniform distribution of volume sources, with a line density $\pm 2\pi c$ (the sign choice is irrelevant) along the axis of symmetry. Subtraction of the solenoidal flux vector $-c^2\hat{\boldsymbol{x}}/\sigma^2$ from the flux vector in (2.10) removes this anomaly for the case of constant swirl circulation.

This modification can be extended to the general case of axisymmetric flow where the swirl circulation $2\pi T$ is not constant. If $2\pi T_0(x)$ is the value of the swirl circulation on the axis of symmetry and $\pm\boldsymbol{u}_{\mathrm{T}}$ is the irrotational velocity field generated by the equivalent source distribution with line density $\pm 2\pi T_0$, then subtraction of the solenoidal flux vector

$$(u_{\mathrm{T}}^2-v_{\mathrm{T}}^2)\hat{\boldsymbol{x}}+2u_{\mathrm{T}}v_{\mathrm{T}}\hat{\boldsymbol{\sigma}} \tag{2.13}$$

provides a conservation principle which removes the effects of constant swirl circulation at $|x|=\infty$ and along the axis of symmetry. This flux vector allows measurement of the axial component of moment of whirl production in swirling flows in terms of production along the axis of symmetry and, if necessary, at $|x| = \infty$. This would not have been possible with (2.10), since the term $-T^2\hat{\boldsymbol{x}}/\sigma^2$ there calls for an infinite flux tangential to the axial swirling cause. It should be noted that, unlike $\boldsymbol{u}_p$, $\boldsymbol{u}_T$ in no way contributes to the axial half-plane velocity $\boldsymbol{u}$.

In general axisymmetric viscous flow the conservation equation for the axial component of moment of whirl with volume density m now becomes

$$\frac{\partial m}{\partial t} + \operatorname{div} \boldsymbol{N} = 0, \tag{2.14}$$

where

$$\boldsymbol{N} = \boldsymbol{N}_A + \boldsymbol{N}_S, \tag{2.15}$$

$$\boldsymbol{N}_A = m(\boldsymbol{u} + 2\boldsymbol{q}_0) - \nu \nabla m + (u^2 - v^2)\hat{\boldsymbol{x}} + 2uv\hat{\boldsymbol{\sigma}} - [(u_p^2 - v_p^2)\hat{\boldsymbol{x}} + 2u_p v_p \hat{\boldsymbol{\sigma}}] \tag{2.16}$$

and

$$\boldsymbol{N}_S = -T\boldsymbol{\Omega} - \frac{T^2}{\sigma^2}\hat{\boldsymbol{x}} - [(u_T^2 - v_T^2)\hat{\boldsymbol{x}} + 2u_T v_T \hat{\boldsymbol{\sigma}}]. \tag{2.17}$$

The axial half-plane flux vector $\boldsymbol{N}$ for the axial component of moment of whirl in swirling flow is thus composed of a contribution $\boldsymbol{N}_A$ resulting from the axial half-plane flow and a contribution $\boldsymbol{N}_S$ that results from the swirling motion.

3. Quantitative characterization of axial causes in conically similar viscous flows with swirl

As described in §2, the conservation principle for the axial component of moment of whirl is needed in conically similar viscous flows to measure, and hence classify, the two independent ring circulation causes. The main requirement of the conservation principle is that, since it concerns conservation of a moment of ring-circulation density $(\sigma^2 l)$, its flux vector should be directly independent of the volume and kinematic swirl angular-momentum singularities.

When swirl is present, the flux vector for the axial component of moment of whirl is $\boldsymbol{N}$ as given in (2.15). This includes the particular case of swirl-free flow described in §4 of I, and decouples the axial swirl circulation cause strengths from the flux of the axial component of moment of whirl. There remains, however, the natural flux $-T\boldsymbol{\Omega}$ of the axial component of moment of whirl, which is merely the σ^2 moment of the flux of ring circulation along the vortex tubes needed to account for rotation and stretching of the vortex tubes. In conically similar flow this term describes a concentrated source of the axial component of moment of whirl at the origin of strength

$$\pi\nu^2[\tau^2(+1) - \tau^2(-1)]. \tag{3.1}$$

The term $-T\boldsymbol{\Omega}$ is solenoidal. In this paper, for the purpose of characterizing the ring circulation singularities, we standardize on measuring the excess rate of discharge of the axial component of moment of whirl over this natural production induced by swirl. This external production of the axial component of moment of whirl is described by the discharge of the flux vector $\boldsymbol{N} + T\boldsymbol{\Omega}$ from the axis. This discharge is described by a flux function X, which may be written as

$$X = X_A + X_S, \tag{3.2}$$

where

$$X_A = \nu^2[\chi_A(\mu) - k_A \ln r] \tag{3.3}$$

and

$$X_S = \nu^2[\chi_S(\mu) - k_S \ln r]. \tag{3.4}$$

X_A and X_S are the flux functions for the axial half-plane and swirl contributions to the flux vector $\boldsymbol{N}+\boldsymbol{T\Omega}$. Thus

$$\boldsymbol{N}+\boldsymbol{T\Omega} = \boldsymbol{N}_A+\boldsymbol{N}_S+\boldsymbol{T\Omega}, \tag{3.5}$$

where

$$\boldsymbol{N}_A = \frac{\nu^2}{r^2}\left(-\chi_A'(\mu)\,\hat{\boldsymbol{r}}+\frac{k_A(\mu)}{(1-\mu)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}}\right) \tag{3.6}$$

and

$$\boldsymbol{N}_S+\boldsymbol{T\Omega} = \frac{\nu^2}{r^2}\left(-\chi_S'(\mu)\,\hat{\boldsymbol{r}}+\frac{k_S(\mu)}{(1-\mu^2)^{\frac{1}{2}}}\hat{\boldsymbol{\theta}}\right). \tag{3.7}$$

From (2.16) and (2.17) it follows that

$$-\chi_A'(\mu) = (3-f')(1-\mu^2)g-\frac{\mu}{1-\mu^2}\{f^2-f_p^2\}-\{f^2-f_p^2\}'+\mu\{(f')^2-(f_p')^2\}, \tag{3.8}$$

$$k_A(\mu) = (1-\mu^2)^2g'-(1-\mu^2)fg-\{f^2-f_p^2\}+\mu\{f^2-f_p^2\}'+(1-\mu^2)\{(f')^2-(f_p')^2\}, \tag{3.9}$$

$$-\chi_S'(\mu) = -\frac{\mu}{1-\mu^2}\{\tau^2-f_T^2\}+\{f_T^2\}'-\mu\{f_T'\}^2 \tag{3.10}$$

and

$$k_S(\mu) = \tau^2+f_T^2-\mu\{f_T^2\}'-(1-\mu^2)\{f_T'\}^2. \tag{3.11}$$

Here $f_p(\pm 1) = f(\pm 1)$, $f_T(\pm 1) = \tau(\pm 1)$, and f_p and f_T are both linear in μ.

As in the swirl-free classification in §4 of I, the inverse-square radial dependence of these terms means that the radial and transverse rates of discharge are independently conserved. That is, there is no interchange of the axial component of moment of whirl between the radial and transverse fluxes, with the result that these discharges classify two independent ring circulation producing singularities.

The strength of the ring circulation producing singularity at the origin, as specified by the radial discharge of $\boldsymbol{N}+\boldsymbol{T\Omega}$, is L, where

$$L = 2\pi\nu^2[\chi_A(-1)+\chi_S(-1)-\chi_A(1)-\chi_S(1)], \tag{3.12}$$

while the strength of the axial component of moment of whirl production associated with an antisymmetric distribution of these point singularities is

$$K = 2\pi\nu^2 k = 2\pi\nu^2(k_A(\mu)+k_S(\mu)). \tag{3.13}$$

Here K/r measures the rate of discharge per unit radial thickness from the right half-axis over spherical surfaces into the left half-axis from a line distribution of singularities on each half-axis whose line density is inversely proportional to the distance from the origin (as conical similarity demands). The constancy of the expression for k is a result of the independent conservation of the transverse rate of discharge of the axial component of moment of whirl, though k may also be viewed as an integral invariant of I (3.5) obtained with the integrating factor $1-\mu^2$.

Equations (3.12) and (3.13) generalize the classification of the ring-circulation singularities, developed in §4 of I, to swirling flow, and describe respectively the strengths of a point source and an antisymmetric conically similar distribution of axial component of moment of whirl sources.

In the general characterization of swirling flows, the strengths of the uniform half-line volume sources are unaffected by the presence of swirl, and they have their strengths in conically similar viscous flows specified, as before, by

$$M_{\pm 1} = \mp 2\pi\nu f(\pm 1). \tag{3.14}$$

The final two conically similar causes are, as dimensional analysis indicates, uniform half-line sources of kinematic swirl angular momentum. The linear radial dependence of the flux function Λ for kinematic swirl angular momentum (I (3.21)) ensures uniformity of the half-line sources. Their independence is ensured by the independence of the values assignable to the swirl circulation about each half-axis when viewed as boundary conditions for I (3.6). The strength of kinematic swirl angular momentum production is then

$$J_{\pm 1} = \mp 2\pi\nu^2 \lambda(\pm 1) \tag{3.15}$$

$$= \mp 2\pi\nu^2 [f(\pm 1) \mp 2] \tau(\pm 1), \tag{3.16}$$

where $2\pi\nu\tau(\pm 1)$ is the swirl circulation about each half-axis. This completes the classification of swirling conically similar viscous flows in terms of axial causes. A list summarizing the formulae measuring individual axial cause strengths in swirling conically similar viscous flows appears in table 1.

A discussion of the axial half-plane jet induced by swirl concludes this section. If

$$\left.\begin{aligned} f_{\mathrm{P}} &= \tfrac{1}{2}\{f(-1)(1-\mu) + f(1)(1+\mu)\} \\ \text{and} \quad f_{\mathrm{T}} &= \tfrac{1}{2}\{\tau(-1)(1-\mu) + \tau(1)(1+\mu)\} \end{aligned}\right\} \tag{3.17}$$

are substituted in (3.9) and (3.11), (3.13) can be rearranged to yield

$$k_{\mathrm{A}}(\mu) = k - k_{\mathrm{S}}(\mu), \tag{3.18}$$

where now

$$\left.\begin{aligned} k_{\mathrm{A}}(\mu) &= (1-\mu^2)^2 g' - (1-\mu^2) f g - f^2 + \mu (f^2)' + (1-\mu^2)(f')^2 + f(-1) f(1) \\ \text{and} \quad k_{\mathrm{S}}(\mu) &= \tau^2 + \tau(-1)\tau(1). \end{aligned}\right\} \tag{3.19}$$

Here k_{A}/r is the contribution per unit radial thickness to the transverse ($\boldsymbol{\hat{\theta}}$) rate of discharge of the axial component of moment of whirl arising from the axial half-plane flow, while k_{S}/r describes those contributions arising from swirl.

In §4 of II it has been seen that if k_{A} is non-zero at a half-axis then there exists a strong axial jet about that half-axis. In (3.18) exactly the same expression $k_{\mathrm{A}}(\mu)$ appears ($k_{\mathrm{A}}(\mu)$ is as given in (3.19)). It therefore follows from (3.18) and (3.19) that swirl produces, at the axis of symmetry, a local reduction of

$$2\pi\nu^2(\tau^2 + \tau(1)\tau(-1)) \tag{3.20}$$

in the strength of the axial distribution of the axial component of moment of whirl sources *apparent* to the axial half-plane flow. If $k - k_{\mathrm{S}}$ is non-zero at a half-axis, the axial half-plane flow locally about that half-axis will experience an axial jet asymptotically equivalent to that of §4 in II with

$$C = \tfrac{1}{4}(k - k_{\mathrm{S}}(\pm 1)). \tag{3.21}$$

It should be noted that in swirling flow $k_{\mathrm{A}}(\mu)$ is not, in general, constant. In contrast with swirl-free flow, the transverse rate of discharge of the axial component of moment of whirl resulting from the axial half-plane flow alone ($k_{\mathrm{A}}(\mu)\,\mathrm{d}r/r$) is not conserved and varies with μ. In swirling flow, transverse conservation of the axial component of moment of whirl has an input from the swirling motion which acts as a potential supplier of the axial component of moment of whirl to the axial half-plane flow. (In this sense ring and swirl circulation become interchangeable forms of the same general quantity: circulation.)

Conservation principle	Volume density flux function	Non-dimensional governing equations	Axial cause(s), strength(s) of production, dimension of production	Expression(s) for cause strength(s) in terms of the non-dimensional conically similar viscous-flow solutions	Relations imposed as a result of specifying the strength of a cause
Volume	1, $\psi = \nu r f(\mu)$	$g = -f''(\mu)$	Uniform half-line source of volume on $\mu = \pm 1$, $M_{\pm 1}$, $\dfrac{\text{volume}}{\text{length} \times \text{time}}$	$M_{\pm 1} = \mp 2\pi\nu f(\pm 1)$	$A + C = [f(1) - f(-1)] - \frac{1}{4}[f^2(1) + f^2(-1)]$, $B = [f(1) + f(-1)] \times [1 - \frac{1}{4}\{f(1) - f(-1)\}]$
Ring circulation	$\dfrac{l}{2\pi} = \dfrac{\nu g(\mu)}{2\pi r^3}$, $\dfrac{-B}{2\pi} = \dfrac{-\nu^2\beta(\mu)}{2\pi r^2}$	$(1-\mu^2)f' + 2\mu f - \frac{1}{2}f^2 = G(\mu, \tau^2) + A\mu^2 + B\mu + C$, (*) where $G(\mu, \tau^2) = -\left(\frac{1-\mu}{2}\right)^2 \int_{-1}^{\mu} \frac{\tau^2}{(1-\xi)^2}\,d\xi - \left(\frac{1+\mu}{2}\right)^2 \int_{\mu}^{1} \frac{\tau^2}{(1+\xi)^2}\,d\xi$	Point source of axial component of moment of whirl, L:	$L = 2\pi\nu^2[\chi(-1) - \chi(1)]$, where $\chi = \int_{\mu}^{1} \left[(3-f')(1-\mu^2)g - \frac{\mu}{1-\mu^2}\{f^2 + \tau^2 - f_p^2 - f_T^2\} - \{f^2 - f_p^2 - f_T^2\}' + \mu\{(f')^2 - (f_p')^2 - (f_T')^2\} \right] d\mu$ and f_p and f_T are given by (3.17);	Determines a particular integral of *;
Axial component of moment of whirl	$m = \sigma^2 l = \frac{\nu}{r}(1-\mu^2)g(\mu)$, $X = \nu^2[\chi(\mu) - k \ln r]$		Antisymmetric distribution of axial component of moment of whirl sources with line density inversely proportional to distance from origin, K. $\dfrac{\text{length}^2 \times \text{circulation}}{\text{time}}$	$K = 2\pi\nu^2 k = 2\pi\nu^2[2(C - A) + f(1)f(-1) + \tau(1)\tau(-1)]$, which results from (3.13), (3.9), (3.11), * and its derivatives	$A - C = -\frac{1}{2}k + \frac{1}{2}f(1)f(-1) + \frac{1}{2}\tau(1)\tau(-1)$
Kinematic swirl angular momentum	$T = \nu\tau(\mu)$, $\Lambda = \nu^2 r\lambda(\mu)$	$(1-\mu^2)\tau'' - f\tau' = 0$	Uniform half-line source of kinematic swirl angular momentum on $\mu = \pm 1$, $J_{\pm 1} = \mp 2\pi\nu^2\lambda(\pm 1)$, $\dfrac{\text{swirl angular momentum}}{\text{mass density} \times \text{length} \times \text{time}}$	$J_{\pm 1} = \mp 2\pi\nu^2[f(\pm 1) \mp 2]\tau(\pm 1)$	$\tau(\pm 1) = \dfrac{\lambda(\pm 1)}{f(\pm 1) \mp 2}$

TABLE 1. Summary of conserved quantities, flux functions and axial causes for swirling conically similar viscous flows

4. The mathematical problem of the flow produced by a uniform half-line source of kinematic swirl angular momentum

This section develops the mathematical equations describing the flow produced by a uniform half-line source of kinematic swirl angular momentum. This source is assumed to occur only on $\mu = 1$. Thus

$$\tau(1) = c, \quad \tau(-1) = 0. \tag{4.1}$$

No volume production occurs on either half-axis; hence

$$f(-1) = f(1) = 0. \tag{4.2}$$

The governing differential equations in their most general form are

$$(1-\mu^2)f' + 2\mu f - \tfrac{1}{2}f^2 = G(\mu, \tau^2) + A\mu^2 + B\mu + C \tag{4.3}$$

and

$$(1-\mu^2)\tau'' - f\tau' = 0, \tag{4.4}$$

where

$$G(\mu, \tau^2) = -\left(\frac{1-\mu}{2}\right)^2 \int_{-1}^{\mu} \frac{\tau^2}{(1-\xi)^2} \mathrm{d}\xi - \left(\frac{1+\mu}{2}\right)^2 \int_{\mu}^{1} \frac{\tau^2}{(1+\xi)^2} \mathrm{d}\xi. \tag{4.5}$$

These are I (3.7), I (3.8) and I (3.10) with $\mu_1 = -1$ and $\mu_0 = 1$.

In this system, the solution f always has

$$\lim_{\mu \to \pm 1} (1-\mu^2) f' = 0, \tag{4.6}$$

and the function $G(\mu, \tau^2)$ has

$$G(\pm 1, \tau^2) = 0, \quad G(\mu, \tau^2) \leqslant 0. \tag{4.7}$$

Thus zero volume production ((4.2)) implies, as in swirl-free flow, that

$$A + C = B = 0. \tag{4.8}$$

The strength K of the antisymmetric distribution of axial component of moment. of whirl sources, as given by (3.13), can be rewritten as

$$K = 2\pi\nu^2[\tau^2 - (1-\mu^2)G'' - 2\mu G' + 2G + 2(C-A) + f_{\mathrm{p}}(1)f_{\mathrm{p}}(-1) + f_{\mathrm{T}}(1)f_{\mathrm{T}}(-1)] \tag{4.9}$$

by successive differentiation of (4.3). Zero strength K thus indicates that

$$C - A = 0, \tag{4.10}$$

once it is realized that

$$\lim_{\mu \to -1} (1-\mu^2) G''(u) = G(-1, \tau^2) = G'(-1, \tau^2) = 0 \tag{4.11}$$

and

$$(1-\mu^2)G'' + 2\mu G' - 2G = \tau^2. \tag{4.12}$$

(The products $f_{\mathrm{p}}(1)f_{\mathrm{p}}(-1)$ and $f_{\mathrm{T}}(1)f_{\mathrm{T}}(-1)$ are both zero because there are no volume sources and no swirl-circulation producing singularities on $\mu = -1$.)

The governing differential equation for the non-dimensional stream function $f(\mu)$ then has the simplest form of coupling possible and is

$$(1-\mu^2)f' + 2\mu f - \tfrac{1}{2}f^2 = G(\mu, \tau^2), \tag{4.13}$$

where $G(\mu, \tau^2)$ is given by (4.5). The non-dimensional swirl circulation τ here satisfies (4.4) and (4.1).

The kinematic swirl angular momentum produced per unit length per unit time on the right half-axis is then $J_1 = 4\pi\nu^2 c$. A point source of the axial component of moment of whirl is the remaining singularity that needs to be specified. The standardization, that the discharge of $\boldsymbol{N}+T\boldsymbol{\Omega}$ from the origin is zero, provides a condition for determining the constant of integration in (4.13) and corresponds to setting $L = 0$ in (3.12), whence

$$c^2 = \int_{-1}^{1}\left[\frac{\mu\{f^2+\tau^2-c^2[\frac{1}{2}(1+\mu)]^2\}}{1-\mu^2} - 6f(\mu) - 2\mu(f'(\mu))^2\right]\mathrm{d}\mu. \tag{4.14}$$

A solution of (4.13) and (4.4) that satisfies (4.1) and the integral condition (4.14) is required. Since there is no apparent substitution that linearizes these equations, §§5 and 6 concentrate on the form of the solution for small and large c respectively.

5. Asymptotic form of the solution for small-strength kinematic swirl angular-momentum production

When the kinematic swirl angular-momentum production is small, so is the non-dimensional swirl circulation c, and a series solution in powers of c can be used to reconstruct the solution. In this situation, the boundary conditions for τ and the quadratic coupling term in (4.13) suggest

$$f(\mu) = c^2 f_0 + c^4 f_1 + \dots \tag{5.1}$$

and

$$\tau(\mu) = c\tau_0 + c^3\tau_1 + \dots . \tag{5.2}$$

Substitution in (4.13) then yields that the first-order terms of the series satisfy

$$(1-\mu^2)f_0' + 2\mu f_0 = G(\mu, \tau_0^2), \tag{5.3}$$

with

$$1 = \int_{-1}^{1}\left[\frac{\mu\{\tau_0^2 - [\frac{1}{2}(1+\mu)]^2\}}{1-\mu^2} - 6f_0\right]\mathrm{d}\mu, \tag{5.4}$$

where

$$\tau_0'' = 0, \quad \tau_0(-1) = 0, \quad \tau_0(1) = 1. \tag{5.5}$$

Once f_0 and τ_0 are determined, the second-order terms are specified by

$$(1-\mu^2)f_1' + 2\mu f_1 = \tfrac{1}{2}f_0^2 + G(\mu, 2\tau_0\tau_1), \tag{5.6}$$

with

$$0 = \int_{-1}^{1}\left[2\mu(f_0'(\mu))^2 + 6f_1 - \frac{\mu\{f_0^2 + 2\tau_0\tau_1\}}{1-\mu^2}\right]\mathrm{d}\mu, \tag{5.7}$$

where

$$(1-\mu^2)\tau_1'' = f_0\tau_0', \quad \tau_1(-1) = 0, \quad \tau_1(1) = 0. \tag{5.8}$$

Integration of (5.3) and (5.5) gives

$$\tau_0(\mu) = \tfrac{1}{2}(1+\mu) \tag{5.9}$$

and

$$f_0(\mu) = \tfrac{1}{2}f_0'(-1)(1-\mu^2) + (1-\mu^2)\int_{-1}^{\mu}\frac{G(\xi, \tau_0^2)}{(1-\xi^2)^2}\mathrm{d}\xi, \tag{5.10}$$

where $f_0'(-1)$ is uniquely determined by

$$-\tfrac{1}{6} = \int_{-1}^{1} f_0(\xi)\,\mathrm{d}\xi. \tag{5.11}$$

No separatrix occurs in this almost-Stokes flow. Fluid is forced to flow away from the uniform half-line kinematic swirl angular-momentum production as a result of

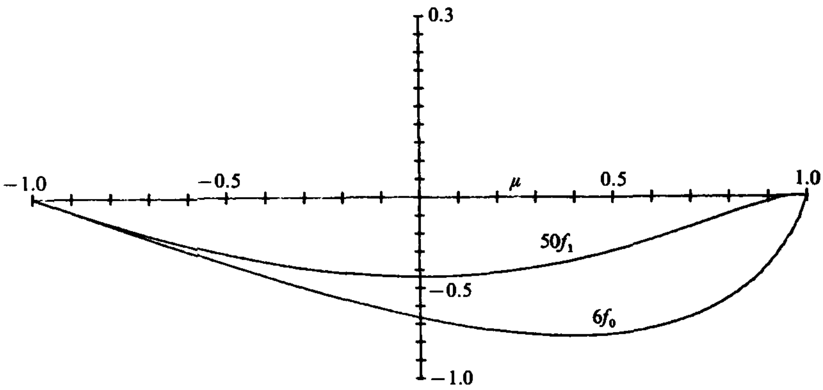

FIGURE 1. The first- and second-order functions in the almost-Stokes flow series expansion of the non-dimensional stream function.

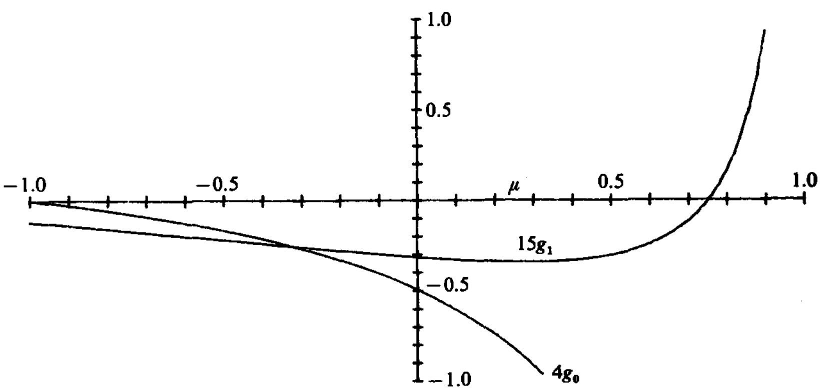

FIGURE 2. The ring circulation distributions for the functions f_0 and f_1 depicted in figure 1.

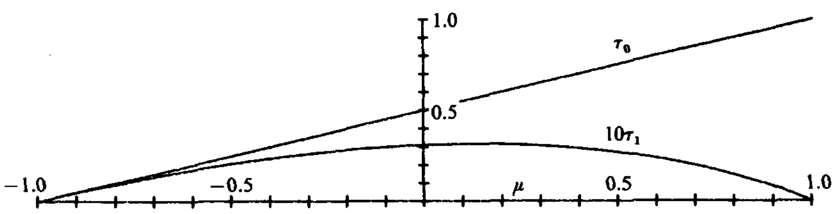

FIGURE 3. The first- and second-order functions in the series expansion for the swirl circulation.

the axial jet surrounding this cause ((3.21) and §4 of II). These results are best proved by evaluating the integrals above, whereby

$$f_0(\mu) = \tfrac{1}{4}(1-\mu)\ln[(\tfrac{1}{2}(1-\mu)] + \tfrac{1}{16}(1-\mu^2) \tag{5.12}$$

and

$$g_0(\mu) = -\frac{1+\mu}{8(1-\mu)}. \tag{5.13}$$

The functions f_0, g_0 and τ_0 are depicted in figures 1, 2 and 3 respectively. The minimum for f_0 occurs for $\mu > 0$, since, by (5.3), $\mu f_0 < 0$ when $f_0' = 0$.

The second-order solutions are now considered. Integration of (5.8) indicates that

$$\tau_1 = \tfrac{1}{8}(1-\mu)\ln[\tfrac{1}{2}(1-\mu)] - \tfrac{1}{64}(1-\mu^2) - \tfrac{1}{8}(1+\mu)\int_0^{\frac{1}{2}(1-\mu)} \frac{\ln\xi}{1-\xi}\,d\xi. \tag{5.14}$$

The second-order swirl circulation field τ_1 then has

$$\tau_1(\mu) \geqslant 0 \tag{5.15}$$

and possesses only a single extremum (a maximum), which occurs for μ positive and less than the zero of $f_0'(\mu)$.

Equation (5.6) can now be integrated to give

$$\frac{f_1}{1-\mu^2} = \tfrac{1}{2}f_1'(-1) + \int_{-1}^{\mu} d\xi \left[\frac{\tfrac{1}{2}f_0^2 + G(\xi, 2\tau_0\tau_1)}{(1-\xi^2)^2}\right], \tag{5.16}$$

which implies that

$$f_1'(1) + f_1'(-1) = -\int_{-1}^{1} d\xi \left[\frac{f_0^2 + 2G(\xi, 2\tau_0\tau_1)}{(1-\xi^2)^2}\right]. \tag{5.17}$$

To reveal the nature of the solution $f_1(\mu)$ it is necessary to determine the sign of the first derivative at $\mu = -1$ that is consistent with zero production of the axial component of moment of whirl.

The integral condition (5.7) yields a second relation between $f_1'(-1)$ and $f_1'(1)$:

$$f_1'(1) - f_1'(-1) = \int_{-1}^{1} \left[(1-\mu^2) f_0' f_0'' - \frac{\mu(2-\mu^2) f_0^2}{(1-\mu^2)^2}\right] d\mu. \tag{5.18}$$

Use of the identity

$$\int_{-1}^{1} \frac{G(\mu, 2\tau_0\tau_1)}{(1-\mu^2)^2}\,d\mu = -\frac{1}{4}\int_{-1}^{1} \frac{\tau_1}{1-\mu}\,d\mu \tag{5.19}$$

and the expressions for f_0 and τ_1 then indicate that

$$\left.\begin{aligned} f_1'(1) + f_1'(-1) &= -\tfrac{1}{128}\{35 + 4\zeta(2) - 32\zeta(3)\} \\ \text{and} \quad f_1'(1) - f_1'(-1) &= -\tfrac{1}{96}\{19 - 3\zeta(2) - 12\zeta(3)\}. \end{aligned}\right\} \tag{5.20}$$

Here $\zeta(z)$ is Riemann's zeta function. This implies that

$$f_1'(-1) \approx -1.4047 \times 10^{-2}, \quad f_1'(1) \approx -1.0297 \times 10^{-2}. \tag{5.21}$$

The functions $f_1(\mu)$ and $f_1''(\mu)$ must therefore change sign at least once. Similar analysis reveals that the functions f_1, f_1'' and f_1''' all have only a single zero.

The functions f_1, g_1 and τ_1 thus appear as simply depicted in figures 1, 2 and 3 respectively. It should be noted that the two-term expansion

$$f = c^2 f_0 + c^4 f_1 \tag{5.22}$$

can be expected to give an accurate representation of the solution only when it has no internal zeros, for, if these occur, then there exists μ_* such that $f(\mu_*) = 0$ with $f'(\mu_*) > 0$, which contradicts $G(\mu_*, \tau^2) \leqslant 0$ in (4.13). Higher-order terms are then necessary to correct the two-term expansion.

The form of the two-term expansion as c increases, however, suggests the presence of a developing internal zero of $f(\mu)$ (a radial jet). This is not really unexpected, since the condition $L = 0$, when large velocities are present, amounts to requiring that there be no dominant left–right preference in the flow. As far as the axial half-plane flow is concerned, for large c, the $L = 0$ condition can be maintained only by an inflow about the swirl-free half-axis to counter the jet ((3.21)) existing about the swirling cause. The matched asymptotic expansions in §6 reveal this feature.

6. Asymptotic form of the solution for large-strength kinematic swirl angular-momentum production

This section develops a matched asymptotic expansion to describe the solution when the swirl circulation strength $2\pi\nu c$ is large. This reveals a well-developed internal boundary layer in the form of a transition between a region of swirl-free potential flow and one of constant swirl circulation ($\tau = c$). The potential flow discharges into a jet lying within the boundary layer. This jet is directed outwards in such a way as to maintain the zero-production condition on L ((4.4)).

When c is large the discussion that concludes §5 indicates that an internal zero of $f(\mu)$ may develop. Suppose then that $f(\mu)$ has an internal zero at $\mu = \mu_*$, where $\lim_{c\to\infty} |\mu_*| < 1$. At $\mu = \mu_*$ (4.13) implies that

$$f'(\mu_*) = \frac{G(\mu_*, \tau^2)}{1-\mu_*^2} < 0, \tag{6.1}$$

while (4.4) gives

$$\tau'(\mu) = D \exp\left[\int_{\mu_*}^{\mu} \frac{f(\xi)}{1-\xi^2}\,\mathrm{d}\xi\right], \tag{6.2}$$

which implies that τ' has a maximum value at $\mu = \mu_*$ and that τ' decays on either side. If c is large the function $G(\mu, \tau^2)$ is then large in magnitude, and the outer solution for $f(\mu)$, from (4.13), is

$$-\tfrac{1}{2}f_0^2 = G(\mu, \tau^2). \tag{6.3}$$

This statement is meaningful, since $G(\mu, \tau^2) \leqslant 0$. It now follows, from (6.2), that τ' decays rapidly and becomes exponentially small outside a small neighbourhood of μ_*. The outer solution for $\tau(\mu)$ is thus

$$\tau_0(\mu) = \begin{cases} 0 & (\mu < \mu_*), \\ c & (\mu > \mu_*). \end{cases} \tag{6.4}$$

Replacement of τ by τ_0 in (6.3) produces only a second-order change in the outer solution for $f(\mu)$. The outer solution for $f(\mu)$ can therefore be taken as

$$f_0 = \begin{cases} [-2G(\mu, \tau_0^2)]^{\frac{1}{2}} & (\mu < \mu_*), \\ -[-2G(\mu, \tau_0^2)]^{\frac{1}{2}} & (\mu > \mu_*). \end{cases} \tag{6.5}$$

The choice of signs in the two regions of definition for f_0 has been made so as to be compatible with $f'(\mu_*) < 0$. Only one internal zero of $f(\mu)$ is possible, since, at any internal zero of f, $f' < 0$. Substitution of τ_0 from (6.4) into the expression for $G(\mu, \tau_0^2)$ in (4.5) then gives

$$f_0(\mu) = \begin{cases} \dfrac{c}{2}\left[\dfrac{1-\mu_*}{1+\mu_*}\right]^{\frac{1}{2}}(1+\mu) & (\mu < \mu_*), \\ -\dfrac{c}{2}\left[\dfrac{(1-3\mu_* + (3-\mu_*)\mu)(1-\mu)}{1-\mu_*}\right]^{\frac{1}{2}} & (\mu > \mu_*). \end{cases} \tag{6.6}$$

On the basis of (6.4) and (6.6), transition expansions are thus necessary for f and

τ about $\mu = \mu_*$. An asymptotic expansion is also needed for $f(\mu)$ near the end of the interval, $\mu = 1$, since the outer solution here fails to satisfy

$$f(\mu) \sim \tfrac{1}{2}G'(1,\tau_0^2)(1-\mu)\ln(1-\mu) \quad \text{as } \mu \to 1.$$

In what follows this expansion will be referred to as the terminating expansion.

By the methods of matched asymptotic expansions, the transition expansion for $f(\mu)$ is

$$f_{\text{tr}}(\xi) = -\tfrac{1}{2}c(1-\mu_*^2)^{\frac{1}{2}} \tanh \frac{\xi}{4(1-\mu_*^2)^{\frac{1}{2}}}, \tag{6.7}$$

and the transition expansion for τ is

$$\tau_{\text{tr}}(\xi) = \frac{c}{2}\left\{\tanh \frac{\xi}{4(1-\mu_*^2)^{\frac{1}{2}}} + 1\right\}, \tag{6.8}$$

where
$$\mu = \mu_* + c^{-1}\xi. \tag{6.9}$$

The terminating expansion for $f(\mu)$ may be expressed in terms of modified Bessel functions, and is

$$f_{\text{te}}(\eta) = -\eta^{\frac{1}{2}} \frac{K_0(\frac{1}{2}\eta^{\frac{1}{2}})}{K_1(\frac{1}{2}\eta^{\frac{1}{2}})}, \tag{6.10}$$

where $\mu = 1 - \eta/c^2$. The μ-widths of the terminating and transition regions are of different orders; however, since the terminating region surrounds the axis of symmetry, their angular widths are both $O(c^{-1})$.

The only undetermined constant in the matched solutions for f and τ is now μ_*. This is specified by the integral condition (4.14), which may be rewritten in terms of the matched expansions for f and τ. Three intermediate variables and inclusion of only the higher-order terms in the individual integrands allow (4.14) to be written in the form

$$\begin{aligned} 0 = c^2 &+ \int_{-1}^{\mu_*-c^{-\delta_1}\eta_1} \mathrm{d}\mu \left[2\mu(f_0')^2 - \frac{\mu}{1-\mu^2}\left[f_0^2 - c^2\left(\frac{1+\mu}{2}\right)^2\right]\right] + \int_{-c^{-\delta_1+1}\eta_1}^{c^{-\delta_2+1}\eta_2} \mathrm{d}\xi\, 2c\mu_*(f_{\text{tr}}'(\xi))^2 \\ &+ \int_{\mu_*+c^{-\delta_2}\eta_2}^{1-c^{-\delta_3}\eta_3} \mathrm{d}\mu \left[2\mu(f_0')^2 - \frac{\mu}{1-\mu^2}\left[f_0^2 + c^2\left[1 - \left(\frac{1+\mu}{2}\right)^2\right]\right]\right] + \int_0^{c^{-\delta_3+2}\eta_3} 2c^2(f_{\text{te}}'(\eta))^2\,\mathrm{d}\eta. \end{aligned} \tag{6.11}$$

The first two terms here are of order c^2 and the transition integral is of order $\mu_* c^3$. The remaining two integrals are collectively of order $c^2 \ln c$ since the second last integral $\sim -\frac{1}{2}c^2 \ln[c^{-\delta_3}\eta_3]$ while the last integral $\sim \frac{1}{2}c^2 \ln[c^{-\delta_3+2}\eta_3]$. The zero of $f(\mu)$, μ_*, is thus determined by

$$2c\mu_* \int_{-\infty}^{\infty} \left(\frac{c}{8}\operatorname{sech}^2 \frac{\xi}{4(1-\mu_*^2)^{\frac{1}{2}}}\right)^2 \mathrm{d}\xi = c^2 \ln c,$$

that is
$$\mu_* \sim -\frac{6 \ln c}{c}. \tag{6.12}$$

Since μ_* does not encroach on $\mu = \pm 1$ as $c \to \infty$, the expansions above are valid. (Implicit in writing transition and terminating expansions for separate regions is the requirement that these regions do not coalesce.) For large c the function f, g and τ thus appear as in figure 4.

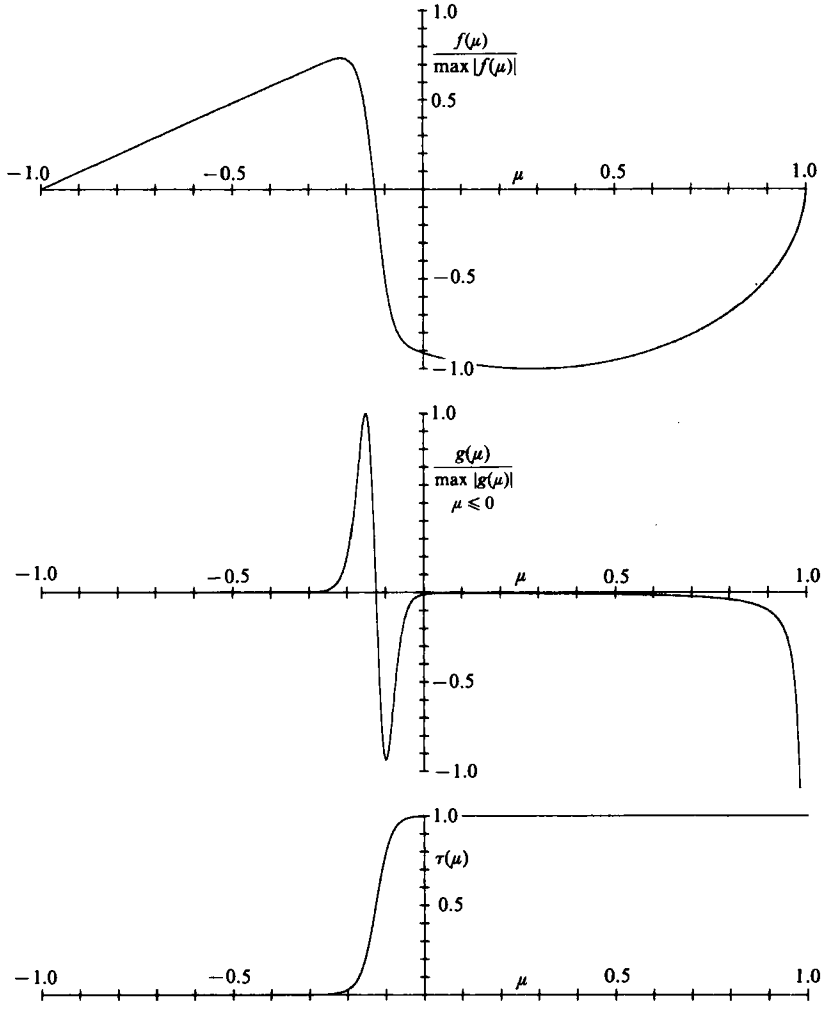

FIGURE 4. The non-dimensional stream function f, swirl circulation τ and ring circulation density g for large-strength kinematic swirl angular momentum production on a half-axis ($J_1 = 4\pi\nu^2 c, c = 100$).

7. Discussion of the flow

Whilst the mathematical description is now complete, it is important to understand the strengths of the individual terms in the flux vectors, since this provides an understanding of the basic local balances for each of the conserved quantities and an alternative check on the validity of the solutions.

For the sake of physical completeness, this section describes in detail the dominant flow features for small and large swirl circulations $2\pi\nu c$ about the uniform half-line source of kinematic swirl angular momentum. The swirling cause is assumed to occupy the right half-axis, with the left half-axis being free of external causes.

The most significant feature for all values of c is that an axial half-plane jet is induced to flow inwards about the swirling cause. This follows from the discussion concluding §3, where swirl was seen locally to reduce the axial component of moment-of-whirl production apparent to the axial half-plane flow. For the problem at hand

$$k = \tau(-1) = f(\pm 1) = 0, \quad \text{and} \quad \tau(1) = c.$$

Expression (3.20) then shows that *the effect of swirl on the axial half-plane flow* is to produce an apparent distribution of the axial component of moment-of-whirl sources with strength

$$K_A = 2\pi\nu^2 k_A(1) = -2\pi\nu^2 c^2 \tag{7.1}$$

on the right half-axis of symmetry only. It is this distribution of sources, apparent to the axial half-plane flow, which drives the inward jet induced by swirl about the swirling cause. The axial half-plane flow here is therefore asymptotically equivalent to that in §4 of II with $C = -\frac{1}{4}c^2$; as $\mu \to 1$

$$(1-\mu^2)f' + f \sim -\tfrac{1}{4}c^2(1-\mu)$$

with

$$f \sim \tfrac{1}{4}c^2(1-\mu)\ln(1-\mu), \quad m \to \frac{\nu c^2}{2r}, \tag{7.2}$$

as (4.13) indicates. Near the swirling cause, conservation (maintenance of the zero transverse flux) of the axial component of moment of whirl is then a result of a balance between the flux terms $m\boldsymbol{q}_0$ and $-2u_T v_T \hat{\boldsymbol{\theta}}$ in the flux vector $\boldsymbol{N}$ ((2.15)). Normal to the swirling half-axis, diffusion then balances fictitious convection of ring circulation. On the free-space half-axis ($\mu = -1$) there is no distribution of the axial component of moment of whirl sources apparent to the axial half-plane flow, since $\tau(-1) = 0$. As a result, the radial velocity here ($f'(-1)$) is finite.

The non-dimensional transverse flux k_S of the axial component of moment of whirl emanates from the swirling cause and diminishes to zero, in driving the axial half-plane flow, by the time the swirl-free half-axis is reached. An equal and opposite flux k_A produced by the axial half-plane flow counters k_S at each location, and thus ensures that the total transverse rate of discharge is zero. Consequently there is no actual production of the axial component of moment of whirl on the half-axes of symmetry.

Even for small swirl circulation strengths, (7.2) results in radial convection of ring circulation being important in a neighbourhood of the swirling cause. For small kinematic swirl angular-momentum production, to first order the flow is almost-Stokes flow. As seen in §3 of II, Landau's point cause then controls the level of ring circulation to first order (the $1-\mu^2$ term in (5.10) and II (3.10)) through its strength L, which specifies the mean axial component of moment of whirl density. (In (5.10) and (5.11)

$$\int_{-1}^{1}(1-\mu^2)g_0(\mu)\,d\mu = \int_{-1}^{1}f_0(\mu)\,d\mu). \tag{7.3}$$

Where no physical causes are distributed along the axis, $L \neq 0$ results in viscous convection of ring circulation into the axis. The absence ($L = 0$) of Landau's cause in the Stokes flow of the current problem would thus, at first order, fix the level of ring circulation so as to avoid viscous convection into the free-space half-axis ($l = 0$ on $\mu = -1$). Negative ring circulation concentrated by the viscous convection field near the right half-axis, diffusion of negative ring circulation from a neighbourhood of the swirling cause and (constant-sign) distributed production of ring circulation by the gradient in the swirl circulation then culminate in ring circulation being distributed with the same sense throughout the Stokes flow ((5.13)). Fluid thus continues flowing in the sense induced when within a neighbourhood of the swirling cause ($f_0(\mu) \leqslant 0$).

In the almost-Stokes flow, kinematic swirl angular momentum diffuses and

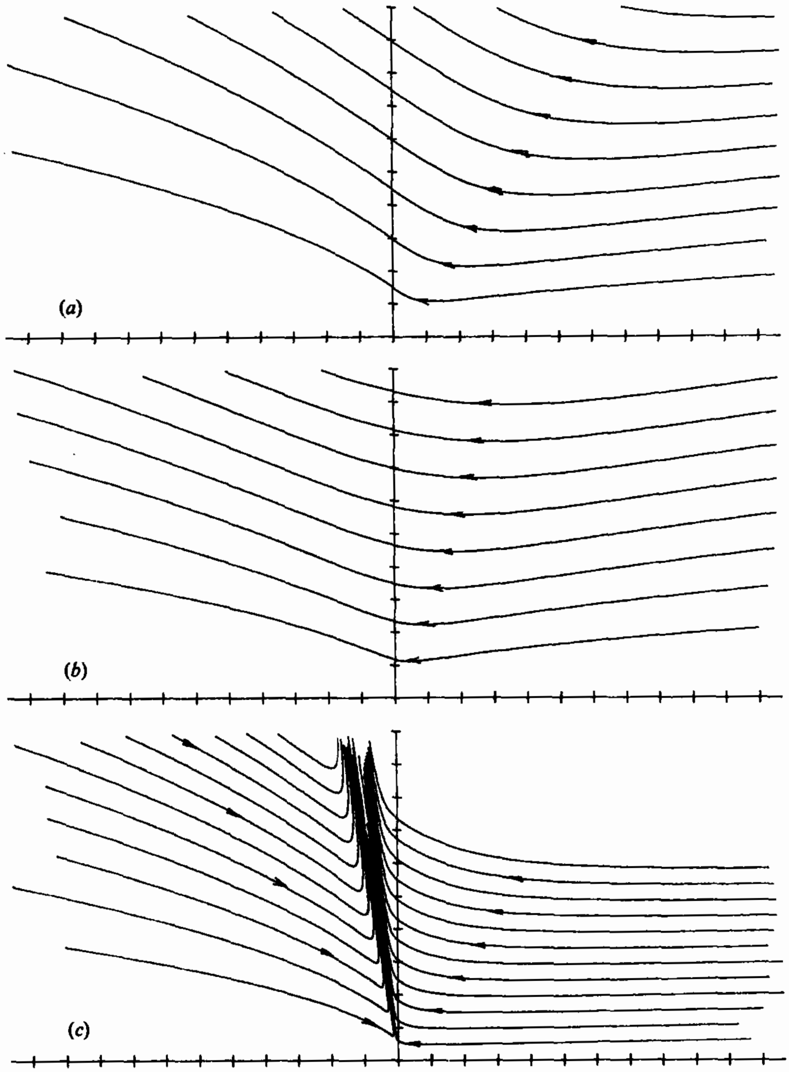

FIGURE 5. Axial half-plane streamlines for the solutions with $J_1 = 4\pi\nu^2 c$ for (*a*) $c = 0.1$; (*b*) 3.0; (*c*) 100. The kinematic swirl angular momentum is produced on the right half-axis of symmetry.

viscously convects so that, on spherical surfaces, swirl circulation is distributed linearly with respect to the axial coordinate ((5.9)). The axial half-plane streamlines then appear as in figure 5 (*a*).

A computed isometric view of the streamlines appears in figure 6 (*a*). (The swirling cause appears on the left at the half-axis $\mu = 1$.) This figure displays the comparative strength of the swirl and axial half-plane velocities. Along the stream surfaces surrounding the swirling cause the swirl circulation is of order c with an axial half-plane velocity of order c^2. Thus for small c the torsion of the streamlines is small ($O(c)$). As the fluid particles spiral along these lines they encounter a progressively weaker swirl circulation field while possessing always the same order axial momentum ($O(c^2)$). The torsion of the streamlines thus increases as the particles move along them. The particles travel roughly parallel to the swirling cause owing to the ring

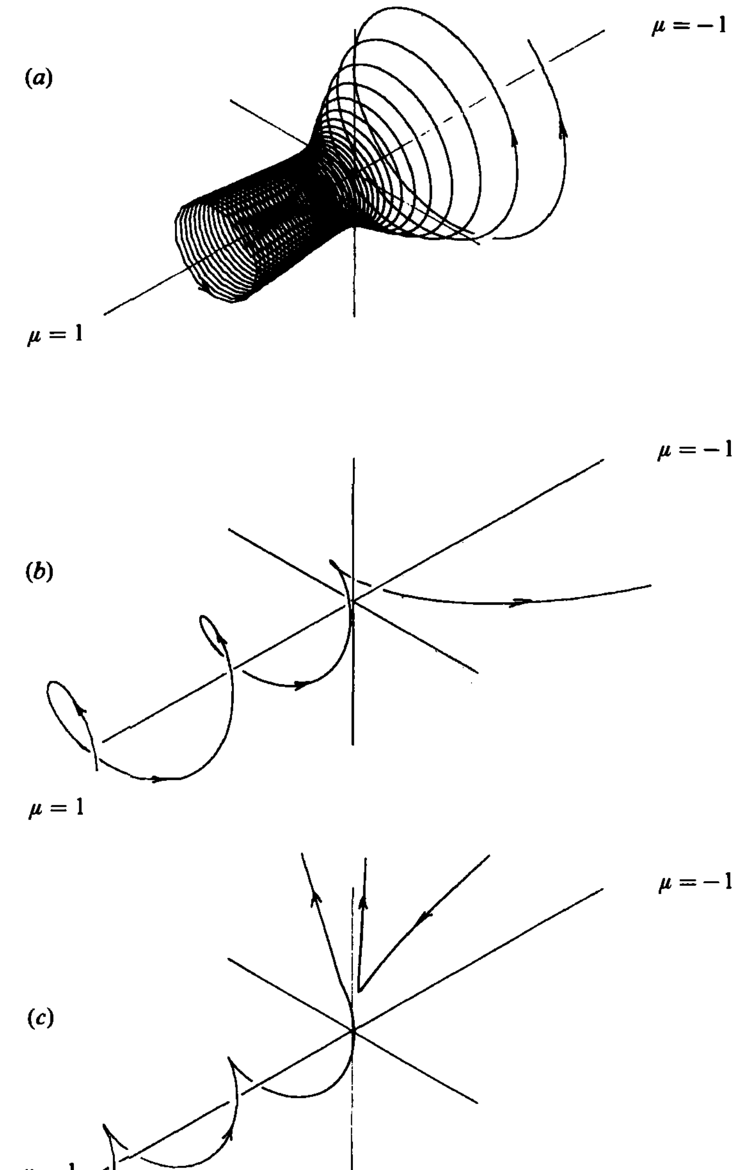

FIGURE 6. An isometric view of the streamlines for the solutions depicted in figure 5: (*a*) $c = 0.1$; (*b*) 3.0; (*c*) 100. Note the comparative strengths of the swirl and axial half-plane velocities.

circulation concentrated there ((5.13)) and spiral outwards from the axis of symmetry after passing the origin owing to the flow becoming more potential in nature ($g_0(-1) = 0$).

As the swirl circulation increases, axial convection becomes important in the previous Stokes-flow region. The induced axial half-plane flow becomes of comparable order to the swirl circulation with the result that the streamlines uncoil (figure 6*b*). If this trend were to continue, kinematic swirl angular momentum would become spread almost uniformly throughout the flow, with the transition to zero swirl circulation at the free space half-axis being achieved in a boundary layer about that axis. The axial half-plane flow would possess a boundary layer at this same location. As the width of this layer diminished, the point source strength of the axial component of moment of whirl would become large and negative and consequently would not satisfy the production requirements of the problem under consideration. The flow configuration of the almost-Stokes flow is thus unacceptable for large kinematic swirl angular-momentum production. A separatrix in the flow must therefore develop as the production of kinematic swirl angular momentum increases. The numerical solutions indicate the critical value of c is very close to 3.

Outflow would occur about this separatrix (6.5) as a result of the strong induced flow that surrounds the swirling cause. Ring circulation and kinematic swirl angular momentum neighbouring the swirling cause, as in the Stokes flow, are thus not confined by this convection field. The region to the right of the separatrix consequently becomes filled with almost-constant swirl circulation ((6.4)). To first order no distributed production of ring circulation then occurs in this region, since there is no gradient in the swirl circulation to rotate the vortex tubes. The axial half-plane flow in this region is then equivalent to having distributed production of the axial component of moment of whirl with a strength $-2\pi\nu^2c^2$ on the right half-axis. This feeds the jet region with half its fluid. The other half is supplied by the flow induced in the region separated from the swirling cause. Convection opposing diffusion of kinematic swirl angular momentum across the jet yields swirl-circulation-free flow about the non-swirling half-axis. No distributed production of ring circulation then occurs here. Convection opposing diffusion of ring circulation across the jet indicates that the flow in the left region is also ring-circulation free. An outer swirl-free potential flow is thus entrained by the jet.

Distributed production of ring circulation occurs only in the jet region, for it is only here that the gradient in the swirl circulation is significant. The distributed production of ring circulation is accounted for by the flux

$$\frac{-T\boldsymbol{\Omega}}{\sigma^2} = -\frac{T}{\sigma^2}\nabla\times\left(\frac{T}{\sigma}\hat{\boldsymbol{\phi}}\right) = \frac{\nu^2\tau\tau'}{r^4(1-\mu^2)}\hat{\boldsymbol{r}} \tag{7.4}$$

of ring circulation (§2 of I). This is directed along the surfaces of constant swirl circulation. Within the jet region in the problem at hand, the radial dependence of this flux indicates that positive ring circulation is generated by rotation of the vortex tubes. Diffusion balances convection of ring circulation across the jet walls. The distributed production of ring circulation in the jet can thus be accounted for only by the radial component of the convective, viscous-convective and diffusive fluxes. Of these the nonlinear convective flux is the dominant one. Thus if the radial velocity at the edge of the jet is of order q_r the distributed ring-circulation production associated with (7.4) is accounted for by having

$$q_r^2 = O(c^2)$$

($\int f'g\,\mathrm{d}\mu = -\frac{1}{2}(f')^2$). The convection field in the outer flow is thus of order c. (The outer axial half-plane flow surrounding the swirling cause is known to be a swirl-free flow produced by a distribution of axial component of moment of whirl sources. In the light of the results of §4 in II then, the strength of the outer flow should be of order c (compare II (4.21) and (6.6) for $\mu > \mu_*$).) The width ϵ of the jet is controlled by the strength of the convection field in the outer flow. Consequently, diffusion opposing convection of ring circulation across the jet region implies

$$\epsilon^{-1} = O(c).$$

The width of the jet region is thus of order c^{-1}. The radial velocity within the jet is then of order c^2 and the intensity of the ring circulation concentrated in this region, as a result of the intense rotation of the vortex tubes, is of order c^3 ((6.7) and (6.9)).

If the rate of production of the axial component of moment of whirl at the origin is to be $\pi\nu^2c^2$ then the non-axial jet must be directed so as to achieve this requirement. The precise condition on the flow is given by (4.14). In sectors of the flow where the radial velocity is large compared with its transverse counterpart, the dominant flux of the axial component of moment of whirl emitted from the origin is seen to be equivalent to that for axial momentum, save a factor of $\frac{1}{2}\rho$. (The dominant term $\mu(f')^2$ appears in the integrand in (4.14).) Consequently, the jet region, which has a radial velocity of order c^2 and a width of order c^{-1}, provides a contribution to the point source strength of axial component of moment of whirl that is of order $\mu_* c^3$, since the momentum emitted radially in the jet is

$$2\pi\nu^2(f'(\mu_*))^2 \times \text{width of jet} = O(c^3),$$

which implies that the axial component of this emitted momentum (axial momentum) is of order $\mu_* c^3$. Here μ_* is the cosine of the angle the jet makes to the swirling cause.

The radial velocity is also large in the jet region about the swirling cause, where the axial velocity has $u \propto \nu c/\sigma$ outside a layer which extends to an angular width of order c^{-1} from the swirling cause (§4 of II). The axial momentum production from this region is thus $O(c^2 \ln c)$ ((6.11)). In the other sectors of the flow the radial and transverse velocities are both of order c. The nonlinear contributions to the point axial component of moment of whirl production are there $O(c^2)$.

The contributions to L from these distinct regions are required, in our problem, to sum to zero. The higher-order terms then indicate that the jet direction μ_* must be such that

$$O(\mu_* c^3) + O(c^2 \ln c) = 0.$$

The jet is thus directed away from the swirling cause at an angle of order $c^{-1} \ln c$ to the plane normal to the axis of symmetry. As the swirl circulation increases, the jet width thus narrows faster than the jet approaches the normal to the axis of symmetry.

The axial half-plane streamlines for a large value of the swirl circulation appear in figure 5(c). In the outer region about the swirling cause the axial half-plane velocity and the swirl circulation are of the same order ((6.4) and (6.6)), and the streamlines are helices with a pitch–circumference ratio of 0.5:

$$u \sim -\frac{c\nu}{2\sigma}, \quad \frac{T}{\sigma} \sim \frac{c\nu}{\sigma}.$$

Fluid flows towards the radial jet, where it is violently ejected with the swirl circulation it obtained from its prior encounter with the swirling cause. In the region separated from the swirling cause the flow is swirl-free potential flow. This fluid is entrained into the jet with no swirl circulation. It gathers kinematic swirl angular momentum when within the jet as a result of diffusion down the gradient from the higher concentrations present in the other region of the flow. The streamlines within the jet have a very small curvature, since the radial velocity here is $O(c)$ larger than the swirl velocity.

REFERENCES

GOL'DSHTIK, M. A. 1960 A paradoxical solution of the Navier–Stokes equations. *Prikl. Math. Mekh.* **24**, 610–621.

HOFFMANN, J. A. 1974 Similarity solutions for the interaction of a potential vortex with free stream sink flow and a stationary surface. *Trans. ASME* I: *J. Fluids Engng* **96**, 49–54.

KIDD, G. J. & FARRIS, G. J. 1968 Potential vortex flow adjacent to a stationary surface. *Trans. ASME* E: *J. Appl. Mech.* **35**, 209–215.

PAULL, R. & PILLOW, A. F. 1985 Conically similar viscous flows. Part 2. One-parameter swirl-free flows. *J. Fluid Mech.* **155**, 343–358.

PILLOW, A. F. & PAULL, R. 1985 Conically similar viscous flows. Part 1. Basic conservation principles and characterization of axial causes in swirl-free flow. *J. Fluid Mech.* **155**, 327–341.

SCHWIDERSKI, E. W. 1969 On the axisymmetric vortex flow over a flat surface. *Trans. ASME* E: *J. Fluid Mech.* **36**, 614–619.

SERRIN, J. 1972 The swirling vortex. *Phil. Trans. R. Soc. Lond.* A **271**, 325–360.

WHANG, Y. C. 1968 Laminar diffusion flame in the core of the vortex. *Phys. Fluids* **11**, 31–37.

WYGNANSKI, I. 1970 Swirling axisymmetrical laminar jet. *Phys. Fluids* **13**, 2455–2460.

YIH, C.-S., WU, F., GARG, A. K. & LEIBOVICH, S. 1982 Conical vortices: a class of exact solutions of the Navier–Stokes equations. *Phys. Fluids* **25**, 2147–2158.